The $\mathcal{L}$-sectional curvature of S-manifolds

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Abstract

We investigate $\mathcal{L}$-sectional curvature of S-manifolds with respect to the Riemannian connection and to certain semi-symmetric metric and non-metric connections naturally related with the structure, obtaining conditions for them to be constant and giving examples of S-manifolds in such conditions. Moreover, we calculate the scalar curvature in all the cases.

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1 Introduction.

In 1963, Yano [13] introduced the notion of f-structure on a C^∞ $(2n+s)$-dimensional manifold M, as a non-vanishing tensor field f of type $(1,1)$ on M which satisfies $f^3+f=0$ and has constant rank $r=2n$. Almost complex $(s=0)$ and almost contact $(s=1)$ are well-known examples of f-structures. The case $s=2$ appeared in the study of hypersurfaces in almost contact manifolds [5, 8] and it motivated that, in 1970, Goldberg and Yano [9] defined globally framed f-structures (also called f.pk-structures), for which the subbundle $\ker f$ is parallelizable. Then, there exists a global frame $\{\xi_1,\dots,\xi_s\}$ for the subbundle

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$\ker f$ (the vector fields $\xi_1, \dots, \xi_s$ are called the structure vector fields), with dual 1-forms $\eta^1, \dots, \eta^s$.

Thus, we can consider a Riemannian metric g on M, associated with a globally framed f-structure, such that $g(fX, fY) = g(X, Y) - \sum_{\alpha=1}^{s} \eta^\alpha(X)\eta^\alpha(Y)$, for any vector fields X, Y in M and then, the structure is called a metric f-structure. Therefore, TM splits into two complementary subbundles $\mathrm{Im} f$ (whose differentiable distribution is usually denoted by $\mathcal{L}$) and $\ker f$ and, moreover, the restriction of f to $\mathrm{Im} f$ determines a complex structure.

A wider class of globally framed f-manifolds (that is, manifolds endowed with a globally framed f-structure) was introduced in [3] by Blair according to the following definition: a metric f-structure is said to be a K-structure if the fundamental 2-form Φ, given by $\Phi(X, Y) = g(X, fY)$, for any vector fields X and Y on M, is closed and the normality condition holds, that is, $[f, f] + 2\sum_{\alpha=1}^{s} d\eta^\alpha \otimes \xi_\alpha = 0$, where $[f, f]$ denotes the Nijenhuis torsion of f. A K-manifold is called an S-manifold if $d\eta^\alpha = \Phi$, for all $\alpha = 1, \dots, s$. If $s = 1$, an S-manifold is a Sasakian manifold. Furthermore, S-manifolds have been studied by several authors (see, for example, [4, 6, 10, 12]).

It is well known that there are not exist S-manifolds ($s \geq 2$) of constant sectional curvature and, for Sasakian manifolds, the unit sphere is the only one. This is due to the fact that $K(X, \xi_\alpha) = 1$ and $K(\xi_\alpha, \xi_\beta) = 0$, for any unit vector field $X \in \mathcal{L}$ and any $\alpha, \beta = 1 \dots, s$. For this reason, it is interesting to study the sectional curvature of planar sections spanned by vector fields of $\mathcal{L}$ (called $\mathcal{L}$-sectional curvature) and to obtain conditions for this sectional curvature to be constant.

Further, in 1924 Friedmann and Schouten [7] introduced semi-symmetric linear connections on a differentiable manifold. Later, Hayden [11] defined the notion of metric connection with torsion on a Riemannian manifold. More precisely, if ∇ is a linear connection in a differentiable manifold M, the torsion tensor T of ∇ is given by $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, for any vector fields X and Y on M. The connection ∇ is said to be symmetric if the torsion tensor T vanishes, otherwise it is said to be non-symmetric. In this case, ∇ is said to be a semi-symmetric connection if $T(X, Y) = \eta(Y)X - \eta(X)Y$, for any X, Y, where η is a 1-form on M. Moreover, if g is a Riemannian metric on M, ∇ is called a metric connection if $\nabla g = 0$, otherwise it is called non-metric. It is well known that the Riemannian connection is the unique metric and symmetric linear connection on a Riemannian manifold. Recently, S-manifolds endowed with a semi-symmetric either metric or non-metric connection naturally related with the S-structure have been studied in [1, 2].

In this paper, we investigate $\mathcal{L}$-sectional curvature of S-manifolds with respect to the Riemannian connection and to the semi-symmetric metric and non-metric connections introduced in [1, 2], obtaining conditions for them to be constant and giving examples of S-manifolds in such conditions. Moreover, we calculate the scalar curvature in all the cases.

2 Preliminaries on S-manifolds.

A $(2n + s)-$ dimensional differentiable manifold M is called a *metric f-manifold* if there exist a $(1, 1)$ type tensor field f, s vector fields $\xi_1, \dots, \xi_s$, called *structure vector fields*, s

1-forms $\eta^1, \dots, \eta^s$ and a Riemannian metric g on M such that

$$f^2 = -I + \sum_{\alpha=1}^{s} \eta^\alpha \otimes \xi_\alpha, \ \eta^\alpha(\xi_\beta) = \delta_{\alpha\beta}, \ f\xi_\alpha = 0, \ \eta^\alpha \circ f = 0, \tag{2.1}$$

$$g(fX, fY) = g(X,Y) - \sum_{\alpha=1}^{s} \eta^\alpha(X)\eta^\alpha(Y), \tag{2.2}$$

for any $X, Y \in \mathcal{X}(M)$, $\alpha, \beta \in \{1, \dots, s\}$. In addition:

$$\eta^\alpha(X) = g(X, \xi_\alpha), \ g(X, fY) = -g(fX, Y). \tag{2.3}$$

Then, a 2-form Φ is defined by $\Phi(X,Y) = g(X, fY)$, for any $X, Y \in \mathcal{X}(M)$, called the *fundamental 2-form*. In what follows, we denote by $\mathcal{M}$ the distribution spanned by the structure vector fields $\xi_1, \dots, \xi_s$ and by $\mathcal{L}$ its orthogonal complementary distribution. Then, $\mathcal{X}(M) = \mathcal{L} \oplus \mathcal{M}$. If $X \in \mathcal{M}$, then $fX = 0$ and if $X \in \mathcal{L}$, then $\eta^\alpha(X) = 0$, for any $\alpha \in \{1, \dots, s\}$, that is, $f^2 X = -X$.

In a metric f-manifold, special local orthonormal basis of vector fields can be considered: let U be a coordinate neighborhood and E_1 a unit vector field on U orthogonal to the structure vector fields. Then, from (2.1)-(2.3), fE_1 is also a unit vector field on U orthogonal to E_1 and the structure vector fields. Next, if it is possible, let E_2 be a unit vector field on U orthogonal to E_1, fE_1 and the structure vector fields and so on. The local orthonormal basis $\{E_1, \dots, E_n, fE_1, \dots, fE_n, \xi_1, \dots, \xi_s\}$,, so obtained is called an *f-basis*.

Moreover, a metric f-manifold is *normal* if

$$[f, f] + 2\sum_{\alpha=1}^{s} d\eta^\alpha \otimes \xi_\alpha = 0,$$

where $[f, f]$ denotes the Nijenhuis tensor field associated to f. A metric f-manifold is said to be an *S-manifold* if it is normal and

$$\eta^1 \wedge \dots \wedge \eta^s \wedge (d\eta^\alpha)^n \neq 0 \text{ and } \Phi = d\eta^\alpha, \ 1 \leq \alpha \leq s.$$

Observe that, if $s = 1$, an S-manifold is a Sasakian manifold. For $s \geq 2$, examples of S-manifolds can be found in [3, 4, 10].

If ∇ is a linear connection on an S-manifold and K denotes the sectional curvature associated with ∇, the *$\mathcal{L}$-sectional curvature* $K_\mathcal{L}$ of ∇ is defined as $K_\mathcal{L}(X,Y) = K(X,Y)$, for any $X, Y \in \mathcal{L}$. The *scalar curvature* of the S-manifold with respect to ∇ is given by

$$\tau = \frac{1}{2} \sum_{i,j=1}^{2n+s} K(e_i, e_j), \tag{2.4}$$

for any local orthonormal frame $\{e_1, \dots, e_{2n+s}\}$ of tangent vector fields to M.

3 The $\mathcal{L}$-sectional curvature of S-manifolds.

From now on, let M denote an S-manifold $(M, f, \xi_1, \ldots, \xi_s, \eta^1, \ldots, \eta^s, g)$ of dimension $2n+s$. We are going to study the sectional curvature of M with respect to different types of connections on M.

3.1 The case of the Riemannian connection.

First, let ∇ denote the Riemannian connection of g. For the sectional curvature K of ∇, in [6] it is proved that

$$K(\xi_\alpha, X) = R(\xi_\alpha, X, X, \xi_\alpha) = g(fX, fX), \tag{3.1}$$

for any $X \in \mathcal{X}(M)$ and $\alpha \in \{1, \ldots, s\}$. Consequently, if $s = 1$, the unit sphere is the only Sasakian manifold of constant (sectional) curvature. If $s \geq 2$, from (3.1), we deduce that M cannot have constant sectional curvature. For this reason, it is necessary to introduce a more restrictive curvature. In general, a plane section π on a metric f-manifold M is said to be an *f-section* if it is determined by a unit vector X, normal to the structure vector fields and fX. The sectional curvature of π is called an *f-sectional curvature*. An S-manifold is said to be an *S-space-form* if it has constant f-sectional curvature c and then, it is denoted by $M(c)$. The curvature tensor field R of $M(c)$ satisfies [12]:

$$\begin{aligned}
R(X,Y,Z,W) = \sum_{\alpha,\beta=1}^{s} \{ & g(fX,fW)\eta^\alpha(Y)\eta^\beta(Z) \\
& - g(fX,fZ)\eta^\alpha(Y)\eta^\beta(W) + g(fY,fZ)\eta^\alpha(X)\eta^\beta(W) \\
& - g(fY,fW)\eta^\alpha(X)\eta^\beta(Z)\} \\
& + \frac{c+3s}{4}\{g(fX,fW)g(fY,fZ) - g(fX,fZ)g(fY,fW)\} \\
& + \frac{c-s}{4}\{\Phi(X,W)\Phi(Y,Z) - \Phi(X,Z)\Phi(Y,W) - 2\Phi(X,Y)\Phi(Z,W)\},
\end{aligned} \tag{3.2}$$

for any $X, Y, Z, W \in \mathcal{X}(M)$.

Therefore, if M is an S-space-form of constant f-sectional curvature c and considering an f-basis, from (3.1) and (3.2), we deduce that the scalar curvature of M with respect to the curvature tensor field of the Riemanian connection ∇ satisfies:

$$\tau = \frac{n(n-1)(c+3s)}{2} + n(c+2s).$$

Now, in view of (3.1) it is interesting to investigate the conditions for $K_{\mathcal{L}}$ to be constant. In this context, we observe that, if $n = 1$, $K_{\mathcal{L}}$ is actually the f-sectional curvature. Moreover, for $n \geq 2$, we can prove the following theorem.

Theorem 3.1. *Let M be a $(2n+s)$-dimensional S-manifold with $n \geq 2$. If the $\mathcal{L}$-sectional curvature $K_{\mathcal{L}}$ with respect to the Riemannian connection ∇ is constant equal to c, then $c = s$. In this case, the scalar curvature of M is:*

$$\tau = ns(2n+1).$$

Proof. It is clear that if $K_{\mathcal{L}}$ is constant equal to c, then M is an S-space-form $M(c)$. Consequently, from (3.2), we have

$$K_{\mathcal{L}}(X,Y) = \frac{c+3s}{4} + \frac{3(c-s)}{4} g(X,fY)^2, \tag{3.3}$$

for any orthonormal vector fields $X, Y \in \mathcal{L}$. Now, since $n \geq 2$, we can choose X and Y such that $g(X,fY) = 0$. Thus, from (3.3) we deduce

$$\frac{c+3s}{4} = c,$$

that is, $c = s$.

Now, considering a local orthonormal frame of tangent vector fields such that $e_{2n+\alpha} = \xi_\alpha$, for any $\alpha = 1, \ldots, s$, since $K(e_i, e_j) = K_{\mathcal{L}}(e_i, e_j) = s$, $i, j = 1, \ldots, 2n$, $i \neq j$, and using (3.1) and (2.4), we get the desired result for the scalar curvature. ☐

By using (3.2) and (3.3), we have:

Corollary 3.2. *Let $M(c)$ be an S-space-form of constant f-sectional curvature c. Then, M is of constant $\mathcal{L}$-sectional curvature (equal to c) if and only if $c = s$*

Example 3.3. Let us consider $\mathbf{R}^{2n+2+(s-1)}$ with coordinates

$$(x_1 \ldots, x_{n+1}, y_1, \ldots, y_{n+1}, z_1, \ldots, z_{s-1})$$

and with its standard S-structure of constant f-sectional curvature $-3(s-1)$, given by (see [10]):

$$\xi_\alpha = 2\frac{\partial}{\partial z_\alpha},\ \eta^\alpha = \frac{1}{2}\left(dz_\alpha - \sum_{i=1}^{n+1} y_i dx_i\right),\ \alpha = 1, \ldots, s-1,$$

$$g = \sum_{\alpha=1}^{s-1} \eta^\alpha \otimes \eta^\alpha + \frac{1}{4}\sum_{i=1}^{n+1}(dx_i \otimes dx_i + dy_i \otimes dy_i),$$

$$fX = \sum_{i=1}^{n+1}(Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i}) + \sum_{\alpha=1}^{s-1}\sum_{i=1}^{n+1} Y_i y_i \frac{\partial}{\partial z_\alpha},$$

where

$$X = \sum_{i=1}^{n+1}(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + \sum_{\alpha=1}^{s-1} Z_\alpha \frac{\partial}{\partial z_\alpha}$$

is any vector field tangent to $\mathbf{R}^{2n+2+(s-1)}$.

Now, let $S^{2n+1}(2)$ be a $(2n+1)$-dimensional ordinary sphere of radius 2 and $M = S^{2n+1}(2) \times \mathbf{R}^{s-1}$ a hypersurface of $\mathbf{R}^{2n+2+(s-1)}$. Let

$$\xi_s = \sum_{i=1}^{n+1}\left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}\right) - \sum_{i=1}^{n+1}\sum_{\alpha=1}^{s-1} y_i^2 \frac{\partial}{\partial z_\alpha}$$

and $\eta^s(X) = g(X, \xi_s)$, for any vector field X tangent to M. Then, if we put

$$\widetilde{\xi}_\alpha = s\xi_\alpha;\ \widetilde{\eta}^\alpha = \frac{1}{s}\eta^\alpha;\ \alpha = 1, \ldots, s;$$

$$\widetilde{f} = f;\ \widetilde{g} = \frac{1}{s}g + \frac{1-s}{s^2}\sum_{\alpha=1}^{s}\eta^\alpha \otimes \eta^\alpha,$$

it is known ([10]) that $(M, \widetilde{f}, \widetilde{\xi}_1, \ldots, \widetilde{\xi}_s, \widetilde{\eta}^1, \ldots, \widetilde{\eta}^s, \widetilde{g})$ is an S-space-form of constant f-sectional curvature $c = s$. Moreover, from (3.2), it is easy to show that the $\mathcal{L}$-sectional curvature $K_{\mathcal{L}}$ is also constant and equal to s.

3.2 The case of a semi-symmetric metric connection.

In [1], a semi-symmetric metric connection on M, naturally related to the S-structure, is defined by

$$\nabla^*_X Y = \nabla_X Y + \sum_{j=1}^{s}\eta^j(Y)X - \sum_{j=1}^{s} g\,(X, Y)\,\xi_j, \tag{3.4}$$

for any $X, Y \in \mathcal{X}(M)$. For the sectional curvature K^* of ∇^*, the following theorem was proved in [1]:

Theorem 3.4. *Let M be an S-manifold. Then, the sectional curvature of ∇^* satisfies*

(i) $K^*(X, Y) = K(X, Y) - s$;

(ii) $K^*(X, \xi_\alpha) = K^*(\xi_\alpha, X) = 2 - s$;

(iii) $K^*(\xi_\alpha, \xi_\beta) = K^*(\xi_\beta, \xi_\alpha) = 2 - s$,

for any orthonormal vector fields $X, Y \in \mathcal{L}$ and $\alpha, \beta \in \{1, \ldots, s\}$, $\alpha \neq \beta$.

Therefore, from Theorem 3.1, if $s \neq 2$, an S-manifold cannot have constant sectional curvature with respect to the semi-symmetric metric connection defined in (3.4). For $s = 2$, $M = S^{2n+1}(2) \times \mathbf{R}$ endowed with the connection ∇^* and the S-structure given in Example 3.3 is an S-manifold of constant sectional curvature (equal to 0) with respect to ∇^*. Moreover, for any s, by using Theorem 3.1 again and (i) of Theorem 3.4, if the $\mathcal{L}$-sectional curvature associated with ∇^* is constant equal to c, then $c = 0$ and examples of such a situation are given in Example 3.3. In this case, the scalar curvature is given by:

$$\tau^* = \frac{(4ns + s(s-1))(2-s)}{2}.$$

Regarding the f-sectional curvature of ∇^*, from Theorem 4.5 in [1], we know that it is constant if and only if the f-sectional curvature associated with the Riemannian connection is constant too. In this case, if c denotes the constant f-sectional curvature of the Riemannian connection, $c-s$ is the constant f-sectional curvature of ∇^*. Furthermore, from (i) of Theorem 3.4 and (3.3) it is easy to show that

$$K^*_{\mathcal{L}}(X, Y) = \frac{c-s}{4}(1 + 3g(X, fY)^2),$$

for any orthonormal vector fields $X, Y \in \mathcal{L}$. Therefore, considering an f-basis, we deduce that the scalar curvature of a $(2n+s)$-dimensional S-manifold of constant f-sectional curvature c with respect to ∇^* satisfies:

$$\tau^* = \frac{n(n+1)(c-s) + (4ns + s(s-1))(2-s)}{2}.$$

3.3 The case of a semi-symmetric non-metric connection.

In [2], a semi-symmetric non-metric connection on M, naturally related to the S-structure, is defined by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sum_{j=1}^{s} \eta^j(Y)X,$$

for any $X, Y \in \mathcal{X}(M)$. To consider the sectional curvature of $\widetilde{\nabla}$ has no sense because $\widetilde{R}(\xi_\alpha, X, X, \xi_\alpha) = 1$, while $\widetilde{R}(X, \xi_\alpha, \xi_\alpha, X) = 2$, for any unit vector field $X \in \mathcal{L}$ and any $\alpha \in \{1, \dots, s\}$ (see [2] for the details). However, for the $\mathcal{L}$-sectional curvature $\widetilde{K}_{\mathcal{L}}$, we have that $\widetilde{K}_{\mathcal{L}}(X, Y) = K_{\mathcal{L}}(X, Y)$, for any orthogonal vector fields $X, Y \in \mathcal{L}$. Consequently, Theorem 3.3 and Example 3.3 can be applied here. In the case of constant $\mathcal{L}$-sectional curvature (equal to s) and since $\widetilde{R}(\xi_\alpha, \xi_\beta, \xi_\beta, \xi_\alpha) = 1$, for any $\alpha, \beta \in \{1, \dots, s\}$, $\alpha \neq \beta$, the scalar curvature is given by:

$$\widetilde{\tau} = 2ns(n+1) + \frac{s(s-1)}{2}.$$

Regarding the f-sectional curvature of $\widetilde{\nabla}$, in [2] it is proved that it is constant if and only if the f-sectional curvature associated with the Riemannian connection is constant too. In this case, both constant are the same and the curvature tensor field of ∇ is completely determined by c. Furthermore, since from (3.3),

$$\widetilde{K}_{\mathcal{L}}(X, Y) = \frac{c+3s}{4} + \frac{3(c-s)}{4} g(X, fY)^2,$$

for any orthonormal vector fields $X, Y \in \mathcal{L}$, considering an f-basis, we deduce that the scalar curvature of a $(2n+s)$-dimensional S-manifold of constant f-sectional curvature c with respect to $\widetilde{\nabla}$ satisfies:

$$\widetilde{\tau} = \frac{n(n+1)(c+3s) + s(s-1)}{2}.$$

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